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Accurate Numerical Evaluation of Modified Struve Functions Occurring in Unsteady Aerodynamics

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A method for evaluating modified Struve functions is described. The method is reasonably efficient, arithmetically stable, and capable of generating results of arbitrary accurary. This means it can be used to compute modified Struve functions to full register accuracy on computers of any word length.

Nomenclature

A_k = Chebyshev coefficient
$\bar{A}_k$ = approximation to A_k
$a_{\nu k}$ = coefficient in ascending series for $\ell_\nu(x)$
$\bar{a}_{\nu k}$ = approximation to $a_{\nu k}$
B_k = recursion coefficient in scheme[1] for simultaneously evaluating Chebyshev polynomials and summing a series
C_{2k+1} = coefficients in asymptotic series for $\ell_\nu(x)$
E = error bound
i = $\sqrt{-1}$
$I_\nu(x)$ = modified Bessell function of the first kind
K = number of terms in asymptotic expansion
$K_\nu(x)$ = modified Bessel function of the second kind
$L_\nu(x)$ = the modified Struve function (usual definition)
$\ell_\nu(x)$ = a modified Struve function, $(\pi/2)x^{-\nu}[I_\nu(x)-L_\nu(x)]$
$\ell_\nu^{(n)}(x)$ = one-parameter rational approximation to $\ell_\nu(x)$
$\ell_\nu^{n,m}(x)$ = two-parameter rational approximation to $\ell_\nu(x)$
N_1 = integral defined by Eq. (1)
N_2 = integral defined by Eq. (2)
n_1 = integral defined by Eq. (5)
n_2 = integral defined by Eq. (6)
$O(x)$ = order symbol [$y=O(x)$ means y/x is bounded]
r = lateral distance from sending point to receiving point
s_1 = lower limit of integration [Eqs. (12, 16, and 19) of Harder and Rodden[2]]
s_2 = upper limit of integration [Eqs. (13, 17, and 20) of Harder and Rodden[2]]
$T_k(u)$ = Chebyshev polynomial of first kind
u = argument of $T_k(u)$ [x mapped into interval $(-1,1)$]
U = freestream velocity
x = argument of Bessel and Struve functions ($x=\omega r/U$)
ϵ = relative error bound
ω = frequency of vibration
$\sim$ = asymptotic to
$\approx$ = approximately equal
$[x]$ = the greatest integer less than or equal to x

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Introduction

THIS paper describes a method for computing some modified Struve functions that occur in unsteady aerodynamics. The method was developed because the series expansions usually used (for example, Ref. 3, Chs. 9 and 12) are not arithmetically stable and hence cannot give full word length accuracy over all argument ranges of interest. The intent was to develop a stable means of generating the functions for small argument by expanding the integrand of a defining integral into a Taylor's series in such a manner that the resulting series for the Struve function would have all positive coefficients. These coefficients were expressed as rather complicated integrals but fortunately satisfy a simple two-term recursion relation. This relation is unstable in the forward direction but is so stable in the backward direction that it can be started with an arbitrary value.

It is possible to start the recursion with a value or values that will match the leading terms of the asymptotic series for the modified Struve function. As a consequence, a stable series that was derived to compute the Struve function for small argument can be used to generate rational approximations that can be used to compute the function stably and accurately everywhere in the right half-plane. This is a very surprising result. The author knows of no other mathematical function that can be computed this way.

The paper describes how these functions occur in unsteady aerodynamics, how the stable rational approximations converging to the modified Struve functions are derived, and how rapidly executing Chebyshev series can be generated therefrom.

Background

Modified Struve functions of orders 0 and 1 occur when computing the kernel of the integral equation relating lift to downwash in unsteady potential flow. The kernel consists of a combination of elementary functions and two integrals,†

$$N_1=r^2\int_{s_1}^{s_2}(r^2+s^2)^{-3/2}e^{-i(\omega/u)s}\mathrm{d}s \tag{1}$$

and

$$N_2=3r^4\int_{s_1}^{s_2}(r^2+s^2)^{-5/2}e^{-i(\omega/u)s}\mathrm{d}s \tag{2}$$

that cannot be expressed in terms of elementary functions. For planar lifting surfaces only N_1 occurs. In subsonic flow the upper limit s_2 equals ∞, and it is customary to divide the interval of integration into two subintervals $(0,\infty)$ and $(0,s_1)$

†Symbols in Eqs. (1) and (2) have the same meaning as in Harder and Rodden.[2]

as in Yates,[4] for example.‡

$$N_1 = n_1 - r^2 \int_0^{s_1} (r^2+s^2)^{-3/2} e^{-i(\omega/u)s} \mathrm{d}s \tag{3}$$

and

$$N_2 = n_2 - 3r^4 \int_0^{s_1} (r^2+s^2)^{-5/2} e^{-i(\omega/u)s} \mathrm{d}s \tag{4}$$

where

$$n_1 = r^2 \int_0^{\infty} (r^2+s^2)^{-3/2} e^{-i(\omega/u)s} \mathrm{d}s \tag{5}$$

and

$$n_2 = 3r^4 \int_0^{\infty} (r^2+s^2)^{-5/2} e^{-i(\omega/u)s} \mathrm{d}s \tag{6}$$

A similar separation is useful in supersonic flow if s_2 is very large.

The $(0,\ s_1)$ integrals N_1 and N_2 can be evaluated numerically and the $(0,\ \infty)$ integrals n_1 and n_2 can be expressed as modified Bessel and Struve functions using Fourier transform pairs 1.3(7) and 2.3(6) of Ref. 5:

$$n_1 = xK_1(x) + ix(\pi/2)\,[I_1(x) - L_{-1}(x)] \tag{7}$$

and

$$n_2 = x^2K_2(x) - ix^2(\pi/2)\,[I_2(x) - L_{-2}(x)] \tag{8}$$

where

$$x = (\omega/U)r \tag{9}$$

The functions n_1 and n_2 can be expressed in terms of functions of order 0 and 1 using the recursion formulas for modified Bessel and Struve functions (Ref. 3, Eqs. 9.6.26 and 12.2.4):

$$n_1 = xK_1(x) - ix + ix^2\ell_1(x) \tag{10}$$

$$n_2 = 2n_1 + x^2K_0(x) + ix - ix^2\ell_0(x) \tag{11}$$

where

$$\ell_0(x) = (\pi/2)\,[I_0(x) - L_0(x)] \tag{12}$$

$$\ell_1(x) = (\pi/2)x^{-1}\,[I_1(x) - L_1(x)] \tag{13}$$

The series methods of computing $I_\nu(x)$ and $L_\nu(x)$ in the mathematical literature, of which Ch. 12 of Ref. 3 is typical, cannot be used for computing $I_\nu(x) - L_\nu(x)$ because of the severe cancellation effect that results from $L_\nu(x)$ being asymptotic to $I_\nu(x)$ as x increases. For this reason the methods used in the aerodynamic literature, such as that of Watkins, Woolston, and Cunningham[6] are based on crude 3- or 4-decimal-place exponential approximations to the algebraic part of the integrand in Eqs. (5) and (6).

In this paper an accurate method for computing $\ell_\nu(x)$ will be described. The method consists of a two-parameter family of rational approximations,

$$\ell_\nu(x) \approx \frac{\displaystyle\sum_{k=0}^{n+2m} a_{\nu k}(x^k/k!)}{\displaystyle\sum_{k=0}^{n+2m+1} (x^k/k!)} \tag{14}$$

‡The integrals in Yates[4] are related to those herein through an integration by parts.

that converge to $\ell_\nu(x)$ as n increases, a rule for computing the coefficients $a_{\nu k}$ and a rule for assigning values to n and m based on the magnitude of x and the accuracy desired.

Integral Representation

The integral representation implied by Eqs. (5) and (6) is not suitable for numerical evaluations because it has infinite limits and an oscillatory integrand. A more useful representation is obtained from the Laplace transform pair 4.3(12) of Ref. 5:

$$\ell_\nu(x) = \int_0^1 e^{-xt}(1-t^2)^{\nu-(1/2)}\mathrm{d}t \tag{15}$$

This integral has finite limits and a positive integrand, and it can be used to generate series expansions valid for small or large arguments.

Series for Small Arguments

If e^{-xt} is expanded into a Taylor's series and integrated termwise in Eq. (15), a series expansion for $\ell_\nu(x)$ is obtained. This series should not used for numerical evaluation because severe cancellation effects occur for positive real x because the terms alternate in sign.

A series of positive terms is obtained by expressing e^{-xt} as $e^{-xt}\cdot e^{x(1-t)}$, expanding $e^{x(1-t)}$ into a Taylor's series in x, substituting into Eq. (15), and integrating termwise.

$$\ell_\nu(x) = e^{-x}\sum_{k=0}^{\infty} a_{\nu k}(x^k/k!) \tag{16}$$

where

$$a_{\nu k} = \int_0^1 (1-t^2)^{\nu-(1/2)}\mathrm{d}t \tag{17}$$

The coefficients $a_{\nu k}$ satisfy a two-term recursion formula that can be derived by integration by parts:

$$a_{\nu,k+1} = (k+1+2\nu)^{-1}\,[(2k+1+2\nu)a_{\nu k} - 1] \tag{18}$$

Since all the $a_{\nu k}$ are positive [as can be seen by inspection of Eq. (17)], the forward recursion formula above is unstable because of the -1 term in the brackets. However, the corresponding backward recursion formula

$$a_{\nu k} = (2k+1+2\nu)^{-1}\,[(k+1+2\nu)a_{\nu,k+1} + 1] \tag{19}$$

is remarkably stable. This is illustrated in Table 1. In the table $\bar{a}_{0,k}$ are approximations to $a_{0,k}$ generated from Eq. (19) starting with $\bar{a}_{0,21} = 0$.

Series for Large Arguments

If τ/x is substituted for t in Eq. (15) and the radical is expanded in powers of τ by Taylor's theorem, one obtains

$$\ell_\nu(x) \sim \sum_{k=0}^{\infty}\binom{k-\nu-(1/2)}{k}\frac{1}{x^{2k+1}}\int_0^{\infty} e^{-\tau}\tau^{2k}\mathrm{d}\tau$$

$$\sim \sum_{K=0}^{\infty}\binom{k-\nu-(1/2)}{k}\frac{(2k)!}{x^{2k+1}}\left[1 - e^{-x}\sum_{\ell=0}^{2k}\frac{x^{2k-\ell}}{(2k-\ell)!}\right] \tag{20}$$

As $x\rightarrow\infty$ in the right half-plane, the bracket term approaches 1 because e^{-x} approaches zero faster than any negative power of x. Thus

$$\ell_\nu(x) \sim \sum_{k=0}^{\infty}\frac{C_{2k+1}}{x^{2k+1}} \quad \text{as } x\rightarrow\infty \text{ for } |\arg x| < \frac{\pi}{2} \tag{21}$$

Table 1 Exact and recursively generated series coefficients

k	$a_{0,k}$	$\bar{a}_{0,k}$
21	0.0455380708	0.0000000000
20	0.0477146216	0.0243902439
19	0.0501100624	0.0381488430
18	0.0527592212	0.0466169734
17	0.0557047423	0.0525458721
16	0.0589994127	0.0573721159
15	0.0627093743	0.0618694792
14	0.0669186419	0.0664842134
13	0.0717355921	0.0715103329
12	0.0773025079	0.0771853731
11	0.0838100041	0.0837488903
10	0.0915195260	0.0914875140
9	0.1007997505	0.1007829021
8	0.1121881032	0.1121791835
7	0.1265003217	0.1264955645
6	0.1450386348	0.1450360732
5	0.1700210735	0.1700196763
4	0.2055672631	0.2055664868
3	0.2603241503	0.2603237068
2	0.3561944902	0.3561942241
1	0.5707963268	0.5707961494
0	1.5707963268	1.5707961494

where

$$C_{2k+1} = \binom{k-\nu-(1/2)}{k}(2k)! \tag{22}$$

A more convenient expression for C_{2k+1} is

$$C_1 = 1 \tag{23}$$

$$C_{2k+1} = (2k-1-2\nu)(2k-1)C_{2k-1} \tag{24}$$

This is particularly convenient if Eq. (21) is written in nested form

$$\ell_\nu(x) \sim \frac{1}{x}\left(1+\frac{1-2\nu}{x^2}\left(1+\frac{(3-2\nu)3}{x^2}\left(1+\cdots\right)\right)\right) \tag{25}$$

Rational Approximations

One-Parameter Representation

Rational approximations to $\ell_\nu(x)$ are constructed by combining the series in positive powers of x, Eq. (16), with that in negative powers of x, Eq. (21). Consider the truncated form of Eq. (16) first. Let

$$\ell_\nu(x) \approx e^{-x}\sum_{k=0}^{n} a_{\nu k}\frac{x^k}{k!} \tag{26}$$

where the $a_{\nu k}$ are computed by assigning a value to $a_{\nu n}$ and then applying Eq. (19) for $k=n-1(-1)0$. Table 1 shows that $a_{\nu n}$ need not be very accurate. Even so, the best approximation to $a_{\nu n}$ that can be very cheaply obtained should be used. If the radical in Eq. (17) is expanded in a Taylor's series, an asymptotic expansion for $a_{\nu n}$ is obtained:

$$a_{\nu n} \sim \frac{1}{n+1} + \frac{1-2\nu}{(n+1)(n+2)(n+3)} + \cdots \tag{27}$$

If the first term of Eq. (27) is used to compute $a_{\nu n}$ and if the e^{-x} factor in Eq. (26) is replaced by its truncated series representation

$$e^{-x} \approx \left[\sum_{k=0}^{n+1}\frac{x^k}{k!}\right]^{-1} \tag{28}$$

a one-parameter family of rational approximations to $\ell_\nu(x)$ is obtained:

$$\ell_\nu^{(n)}(x) = \frac{\sum_{k=0}^{n} a_{\nu k}(x^k/k!)}{\sum_{k=0}^{n+1}(x^k/k!)} \tag{29}$$

where

$$a_{\nu n} = 1/(n+1) \tag{30}$$

and the other $a_{\nu k}$ for $k=n-1\ (-1)0$ are obtained from Eq. (19).

The function $\ell_\nu^{(n)}(x)$ defined by Eqs. (29, 30, and 19) has several interesting properties:

1) $\ell_\nu^{(n)}(x)$ converges to $\ell_\nu(x)$ as $n\to\infty$ for all finite x.

2) $\ell_\nu^{(n)}(x)$ is asymptotic to $\ell_\nu(x)$ as $x\to\infty$ for all x if $|\arg x| < \pi/2$.

3) Equations (29, 30, and 19) are arithmetically stable if $|\arg x| < \pi/4$.

Thus $\ell_\nu^{(n)}(x)$ can be used to approximate $\ell_\nu(x)$ whenever either x is small or $|\arg x| < \pi/4$. The reason $\ell_\nu^{(n)}(x)$ is asymptotic to $\ell_\nu(x)$ is because

$$a_{\nu n}\frac{(x^n/n!)}{x^{n+1}/(n+1)!}$$

equals the first term of the asymptotic series, Eq. (21).

Two-Parameter Representation

Even better approximations to $\ell_\nu(x)$ for large x are obtained by matching additional terms of Eq. (21). This leads to a two-parameter family of rational approximations to $\ell_\nu(x)$ given by

$$\ell_\nu^{n,m}(x) = \frac{\sum_{k=0}^{n+2m} a_{\nu k}x^k/k!}{\sum_{k=0}^{n+2m+1}(x^k/k!)} \tag{31}$$

where

$$a_{\nu k} = \sum_{\ell=0}^{L}\frac{k!C_{2\ell+1}}{(k+2\ell+1)!} \quad \text{for } k=n+2m(-1)n \tag{32}$$

$$L = [(n+2m-k)/2] \tag{33}$$

and $a_{\nu k}$ for $k=n-1(-1)0$ are obtained from Eq. (19).

Note that $[x]$ means the greatest integer less than or equal to x. The $C_{2\ell+1}$ appearing in Eq. (32) can be computed from either Eq. (22) or Eqs. (23) and (24). Equation (32) can also be written in nested form:

$$a_{\nu k} = \frac{1}{k+1}\left[1+\frac{(1-2\nu)}{(k+2)(k+3)}\left[1+\frac{(3-2\nu)3}{(k+4)(k+5)} + \left[1+\cdots\right]\right]\right] \tag{34}$$

The rational approximations $\ell_\nu^{n,m}(x)$ are defined so that an odd number of asymptotic terms are matched. This is because it was found that matching an even number of asymptotic terms required more work and was less accurate than matching the next lower odd number of terms.

The two-parameter rational approximations $\ell_\nu^{n,m}(x)$ have essentially the same properties as the one-parameter ap-

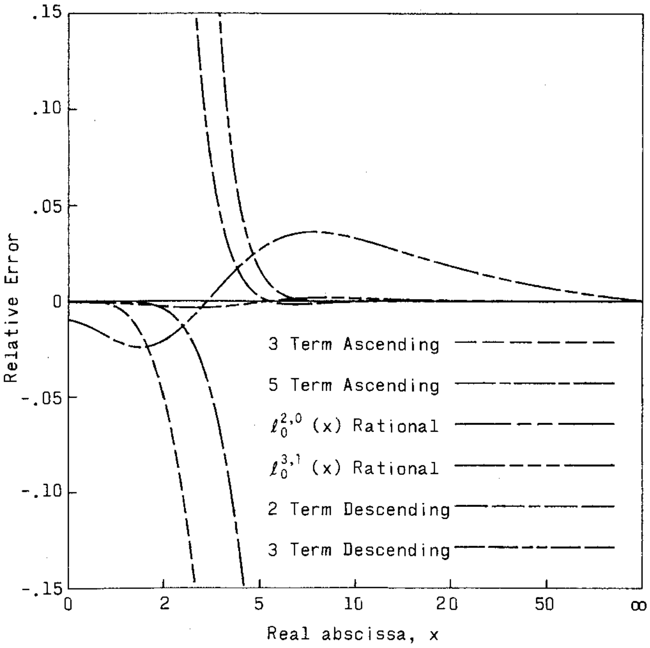

Fig. 1 Relative errors of various approximations to the modified Struve function $\ell_0(x)$.

proximations $\ell_\nu^{(n)}(x)$. The only significant difference is that

$$\ell_\nu^{n,m}(x) - \ell_\nu(x) = 0(x^{-2m-1}) \text{ as } x \to \infty \tag{35}$$

The behavior of the various approximations to $\ell_0(x)$ considered herein is illustrated in Fig. 1. The relative errors are plotted for positive real x. The ascending approximations are

$$\ell_0(x) \approx \sum_{k=0}^{n} a_{0k}(x^k/k!) \tag{36}$$

for $n=3$ and $n=5$. The rational approximations are

$$\ell_0(x) \approx \ell_0^{n,m}(x) \tag{37}$$

with $n=2$, $m=0$, and $n=3$, $m=1$. The descending approximations are

$$\ell_0(x) \approx \sum_{k=0}^{K-1} C_{2k+1}/x^{2k+1} \tag{38}$$

for $K=2$ and $K=3$. For each curve the plotted ordinate is

$$\text{relative error} = \frac{\text{approximation} - \text{true value}}{\text{true value}} \tag{39}$$

The abscissa x has been bilinearly scaled to map the interval $(0, \infty)$ into a finite interval. Note that the a_{0k} used in Eq. (36) are exact values, whereas the a_{0k} used in $\ell_0^{n,m}(x)$ were approximations to the integral (17) generated by Eqs. (19) and (32). This is why Eq. (36) is exact at $x=0$ and $\ell_0{}^{n,m}(x)$ is not. The figure shows that ascending approximations are both very inefficient for $2<x<5$. However, a rational approximation with as few terms as $\ell_0^{3,1}(x)$ has a relative error less than 0.004 over the whole range $(0, \infty)$.

Accuracy Considerations

Four methods of approximations $\ell_\nu(x)$ have been considered: a truncated ascending series, Eq. (16); a truncated descending series, Eq. (21); and two rational approximations, Eqs. (29, 30, 19) and Eqs. (31, 32, 33, 19). The one-parameter rational approximation $\ell_\nu^{(n)}(x)$ need not be considered any further because it is just a special case ($m=0$) of $\ell_\nu^{n,m}(x)$. The truncated ascending series [Eq. (16)] can also be ignored because it has the same range of x values and is less efficient than $\ell_\nu^{n,m}(x)$. At first glance it appears that the descending series (21) can also be ignored because $\ell_\nu^{n,m}(x)$ is a very good approximation to $\ell_\nu(x)$ for large x. This is not true, however, for two reasons. First, the denominator of $\ell_\nu^{n,m}(x)$ is an approximation to e^x, and this overflows a computer's floating point number representation for even moderately large x. Second, the truncated form of (21) is more economical than $\ell_\nu^{n,m}(x)$ for very large x.

Suppose that one wishes to compute $\ell_\nu(x)$ with a relative error $\leq \epsilon$. (On a computer with a floating point mantissa W bits, long ϵ should be 2^{-W}.) Let x_A be the smallest value of x for which the asymptotic series (21) can be used and let x_0 be the smallest value of x for which one term of the asymptotic series [Eq. (21)] can be used. It will be shown later that

$$x_A = \ell n(1/\epsilon) + \ell n(6) \tag{40}$$

and

$$x_0 = \epsilon^{-1/2} \tag{41}$$

(For example if $W=48$ bits, then $x_A = 35$ and $x_0 = 1.7 \times 10^7$.) The following scheme for computing $\ell_\nu(x)$ is recommended.

1) One term asymptotic: If $x > x_0$, then

$$\ell_\nu(x) = 1/x \tag{42}$$

2) K term asymptotic: If $x > x_A$ and $x \leq x_0$, then

$$\ell_\nu(x) = \sum_{k=0}^{K-1} C_{2k-1}/x^{2k+1} \tag{43}$$

where

$$K = \left[\frac{1}{2} \cdot \frac{x_A - 1}{1 + \ell n(x/x_A)}\right] \tag{44}$$

and C_{2k+1} is given by Eqs. (23 and 24).

3) Rational approximation: If $x \leq x_A$, then

$$\ell_\nu(x) = \ell_\nu^{n,m}(x) \tag{45}$$

where

$$n = [x_A + 1] \tag{46}$$

$$m = [(x/2) + 1] \tag{47}$$

and $\ell_\nu^{n,m}(x)$ is computed from Eqs. (31-33) and (19).

The above scheme for computing $\ell_\nu(x)$ required that x_A, x_0, K, n, and m be assigned numerical values depending on ϵ and x. Equations (40, 41, 44, 46, and 47), which assign these values, were developed assuming $\nu = 0$ and x real, and hence are only valid for small ν and $|\arg x| < \pi/4$. If x is complex, then Re $\{x\}$ should be substituted for x in any tests (such as $x < x_A$) and in any expressions that return integer results such as Eq. (47). A discussion of Eqs. (40, 41, 44, 46, and 47) follows.

Consider the truncated series

$$\ell_\nu(x) = \sum_{k=0}^{K-1} \frac{C_{2k+1}}{x^{2K+1}} + \theta_K \frac{C_{2K+1}}{x^{2K+1}} \tag{48}$$

Restrictions must be placed on x and K so that the error term $\theta_K C_{2K+1}/x^{2K+1}$ does not exceed ϵ. For many asymptotic series of this sort (usually those whose terms alternate in sign) it can be shown that $0 < \theta_K < 1$. That is not true in this case because all terms are positive. However, even for series of positive terms θ_K is bounded by some constant, and experience with similar series suggests that this constant is approximately 3. It will be assumed that

$$0<\theta_K<3 \tag{49}$$

The error in Eq. (48) if the sum is truncated at $k=K-1$ is bounded by

$$E=3(C_{2k+1}/x^{2k+1}) \tag{50}$$

If $\nu=0$, then C_{2k+1} simplifies to

$$C_{2k+1}=(2^y/2\pi)[\Gamma(y)]^2 \tag{51}$$

where $y=2K+1$. From Stirling's formula

$$C_{2k+1}\sim(2/y)(y/e)^y \tag{52}$$

so

$$E\sim(6/y)(y/ex)^y \tag{53}$$

This is converted to a relative error ϵ by dividing $\ell_\nu(x)\sim 1/x$; so

$$\epsilon\sim 6(x/y)(y/ex)^y \tag{54}$$

As k increases, the terms of Eq. (48) increase until $2k+1=x$. Then they start decreasing, so x_A can be obtained by solving Eq. (54) with

$$x_A=x=y \tag{55}$$

This gives

$$X_A=\ell n(1/\epsilon)+\ell n 6 \tag{56}$$

as was stated in Eq. (40).

The number of terms K is obtained by solving Eq. (54) for y for arbitrary $x>x_A$. Equation (54) can be written

$$(x/y)(y/ex)^x=e^{-x_A} \tag{57}$$

which rearranges to

$$y=1+\frac{x_A-1}{1+\ell n(x/y)} \tag{58}$$

Equation (58) can be considered an iteration formula for y. Performing one iteration starting with $y=x_A$ gives

$$y\approx 1+\frac{x_A-1}{1+\ell n(x/x_A)} \tag{59}$$

and hence

$$K=\left[\frac{1}{2}\,\frac{x_A-1}{1+\ell n(x/x_A)}\right] \tag{60}$$

Equation (60) overestimates K, but this does not affect the accuracy of the sum [Eq. (48)]. Both Eqs. (56 and 60) were derived assuming $\nu=0$. They are valid for small ν.

Equations (46 and 47) are empirical. They were obtained by varying n,m and comparing errors for $\nu=0$ and $\nu=1$.

A Short Table of Struve Functions

Table 2 lists modified Struve functions of orders 0 and 1 for widely spaced arguments to 28 decimal places. The table is intended for use in certifying computer subprograms.

High-Speed Evaluation of Struve Functions

The rational approximation $\ell_\nu^{n,m}(x)$, together with the asymptotic series, provides a way of computing $\ell_\nu(x)$ to arbitrary accuracy. This method is not suitable for computer use if very high computing speed is required. Fortunately, in unsteady aerodynamics it is almost always possible to organize the calculations so that n_1 and n_2 [Eqs. (5 and 6)] only have to be evaluated once per quadrature chord. In this case the rational and asymptotic approximations are fast enough.

If the calculations cannot be organized in this manner, then a faster scheme is required. A simple way to obtain a faster method is to express $\ell_\nu(x)$ as a Chebyshev series and use a Fast Fourier Transform (FFT) subprogram to compute the Chebyshev coefficients. The input to the FFT subprogram can be computed by the rational and asymptotic approximations previously discussed.

A Chebyshev series is a series of the form

$$f(u)=\sum_{k=0}^{\infty}{}' A_k T_k(u) \tag{61}$$

where the functions $T_k(u)$ are Chebyshev polynomials of the first kind. A prime (or a double prime) on the summation sign

Table 2 A short table of modified Struve functions

x	$\ell_0(x)$	$\ell_1(x)$
0.000	1.5707963267948966192313216915	0.7853981633974483096156608465
0.001	1.5697967193828917460932087926	0.7850649282166672687217429308
0.002	1.5687978967027270821421501599	0.7847318892521509934303135440
0.005	1.5658061303983761280818328733	0.7837339483248191340474695017
0.010	1.5608354858369485430162232433	0.7820746253597779695645807768
0.020	1.5509525214514993321730373286	0.7787705895136459023339811614
0.050	1.5217643376300266370291883422	0.7689741812483230825551506095
0.100	1.4746146170983932837404741767	0.7530247583427005474886565665
0.200	1.3856532913588460732711589271	0.7224870569079129257926681252
0.500	1.1564872837817540954310605088	0.6407344905700246795692427912
1.000	0.8730842426508675390748399694	0.5315491877957080258903182661
2.000	0.5374503890637328028623769390	0.3831777525972875846524765599
5.000	0.2104155460772517638529279915	0.1908286196588690153841679260
10.000	0.1011264407013166543368250964	0.0989630734258619866033006566
20.000	0.0501280166586693236625035664	0.0498740235769744907859086266
50.000	0.0200080290938372030456967523	0.0199919903415724239538086621
100.000	0.0100010009022611154017449244	0.0099989996995484149778344075
200.000	0.0050001250281425997019403261	0.0049998749906214812939711199
500.000	0.0020000080002880288056466298	0.0019999919999039942391933967

means that the first term (or first and last terms) of the sum are to be multiplied by one-half.

A truncated Chebyshev series can be used to approximate $f(u)$ for $-1 \le u \le 1$. To approximate a function such as $\ell_\nu(x)$ with a Chebyshev series, it is necessary to map some interval of interest into $(-1,1)$. It is convenient to use a bilinear transformation for this, mapping x_1 into -1, x_3 into $+1$, and some intermediate point x_2 into 0. That is

$$x = \frac{au+b}{cu+d} \quad \text{and} \quad u = \frac{dx-b}{cx+a} \tag{62}$$

where a, b, c, d are computed from

$$f = (x_3 - x_2)^{-1} \qquad e = (x_2 - x_1)^{-1} \qquad d = f + e$$
$$c = f - e \qquad b = x_2 d \qquad a = 2 + x_2 c \tag{63}$$

The $f(u)$ appearing in Eq. (61) is

$$f(u) = \ell_\nu[(au+b)/(cu+d)] \tag{64}$$

The A_k appearing in Eq. (61) are given by

$$A_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(u)T_k(u)}{\sqrt{1-u^2}}\,\mathrm{d}u \tag{65}$$

This can be written

$$A_k = 2/\pi \int_0^\pi f(\cos\theta)\cos k\theta \,\mathrm{d}\theta \tag{66}$$

by letting $u = \cos\theta$, or it can be written

$$A_k = 1/\pi \int_0^{2\pi} f(\cos\theta)\cos k\theta \,\mathrm{d}\theta \tag{67}$$

by noting that $\cos k\theta$ is even and periodic.

Equation (67) has to be integrated numerically. Let $\bar{A}_k$ be the N-strip trapezoidal approximation to A_k.

$$A_k = (1/\pi) \sum_{\ell=0}^{N}{}'' f(\cos\theta_\ell)\cos k\theta_\ell \cdot 2\pi/N \tag{68}$$

where

$$\theta_\ell = 2\pi\ell/N \tag{69}$$

Since $\theta_N = \theta_0$, this can be written

$$\bar{A}_k = (2/N) \sum_{\ell=0}^{N-1} f_\ell \cos(2\pi k\ell/N) \tag{70}$$

where

$$f_\ell = \ell_\nu\left(\frac{a\cos(2\pi\ell/N) + b}{c\cos(2\pi\ell/N) + d}\right) \tag{71}$$

for

$$\ell = 0(1)(N/2)$$

and

$$f_\ell = f_{N-\ell} \tag{72}$$

for $\ell = (N/2) + 1(1)N-1$.

A very efficient way to implement Eq. (70) for $k = 0\,(1)N-1$ is to use an FFT subprogram to compute

$$F_k = \sum_{\ell=0}^{N-1} f_\ell e^{-i2\pi k\ell/N} \tag{73}$$

Then

$$A_k \approx \bar{A}_k = (2/N)F_k \tag{74}$$

for $k = 0\,(1)(N/2)$. Since the vector f_ℓ is real and even, F_k is also a real even vector. Hence, for $k > N/2$, $\bar{A}_k$ is not a good approximation to A_k. For $k \le N/2$ the relation between $\bar{A}_k$ and A_k is shown by Clenshaw[1] to be

$$\bar{A}_k = A_k + A_{N-k} + A_{N+k} + A_{2N-k} + \cdots \tag{75}$$

For sufficiently continuous $f(u)$, the A_k rapidly approach zero with increasing k; so $\bar{A}_k$ is an excellent approximation to A_k for $k < N/2$.

After the $\bar{A}_k$ have been computed, they must be tested against the error tolerance ϵ. That is, $|\bar{A}_k|$ is compared with ϵ for $k = (N/2) - 1\,(-1)0$ to find the largest n for which

$$|\bar{A}_k| < \epsilon \quad \text{for all } k > n \tag{76}$$

(This implies that N should have been initially set to a value larger than twice the anticipated value of n.) If such an n cannot be found, a new $\bar{A}_k$ vector has to be computed from Eq. (70) by doubling N. If an n is then found,

$$\ell_\nu(x) = \sum_{k=0}^{n}{}' \bar{A}_k T_k\left(\frac{dx-b}{-cx+a}\right) \tag{77}$$

Equation (77) should be evaluated by simultaneously computing the T_k recursively and summing the series as shown by Clenshaw.[1] The procedure is as follows. Let

$$w = 2(dx-b)/(-cx+a) \tag{78}$$

and

$$B_{n+1} = B_{n+2} = 0 \tag{79}$$

Then compute

$$B_k = wB_{k+1} - B_{k+2} + \bar{A}_k \tag{80}$$

Table 3 Chebyshev coefficients for $\ell_1(x)$

k	(0,10)	(10,∞)
0	0.662066488982266	0.082354850541175
1	−0.314064391910733	−0.048121221095341
2	0.105101424991897	0.008088160249652
3	−0.028199678793424	−0.001327610681082
4	0.005940131356411	0.000211392379883
5	−0.000946872057653	−0.000032174087284
6	0.000105871924112	0.000004525759440
7	−0.000006645239221	−0.000000537288805
8	−0.000000053042460	0.000000036074204
9	0.000000043277129	0.000000005854125
10	−0.000000001320223	−0.000000003010591
11	−0.000000000263866	0.000000000617801
12	0.000000000010827	−0.000000000019628
13	0.000000000001993	−0.000000000032387
14	−0.000000000000043	0.000000000010162
15	−0.000000000000016	−0.000000000000123
16	−0.000000000000000	−0.000000000000818
17	0.000000000000000	0.000000000000191
18	0.000000000000000	0.000000000000039
19	0.000000000000000	−0.000000000000027

for $k=n\,(-1)0$. Finally

$$\ell_\nu(x) = \tfrac{1}{2}(B_0 - B_2) \qquad (81)$$

It can be seen that Eqs. (78-81) provide a very fast way of evaluating $\ell_\nu(x)$ provided n is small.

Table 3 gives the Chebyshev coefficients for $\ell_1(x)$ over the intervals (0,10) and $(10,\infty)$. For each interval ϵ was taken to be 10^{-14}, and N was set to 128. For the interval (0,10) x_2 was set to 4 and for $(10,\infty)$ x_2 was set to 30. Thus for the (0,10) interval the w in Eq. (78) is

$$w = 10\text{-}240/(x+20) \qquad (82)$$

and for the $(10,\infty)$ interval

$$w = 2\text{-}80/(x+10) \qquad (83)$$

Concluding Remarks

A method for computing the modified Struve functions which occur in unsteady aerodynamics was developed. The method uses a rational approximation supplemented by an asymptotic series for large argument. Simple recursive formulas for generating the coefficients were described. The method is capable of generating results of arbitrary accuracy. The method can also be used for complex argument and order.

Rational approximations and asymptotic series as described herein are reasonably fast to evaluate. If greater computing speed is needed, Chebyshev series are faster by about a factor of 4. The latter part of this paper described a method for using the rational and asymptotic approximations to generate Chebyshev coefficients.

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